CRITICAL PARAMETERS OF SHOCK WAVES IN A PLASMA

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The critical parameters at which an isomagnetic compression shock occurs are determined using a regular method based on an analysis of the type of singular points of the equations for the structure of standing shock waves propagating at an arbitrary angle to the undisturbed magnetic field in a plasma with finite conductivity and thermal conductivity.

The question of the existence and structure of shock waves in a plasma containing a magnetic field has been studied intensively by many authors [1-10]. In [1] the Hugoniot equations are generalized to the case of magnetogasdynamics and in [2-7] a study is made of these equations, as a result of which shock waves are divided into slow ($M_s \lesssim M \lesssim 1$), intermediate ($1 \lesssim M \lesssim M_f$), and fast ($M \gtrsim M_f$), where M is the velocity of the shock wave with respect to the Alfvén velocity, i.e., the Alfvén–Mach number; M_s is the slow magnetosonic velocity; M_f is the fast magnetosonic velocity. The theory of shock waves in a rarefied plasma was founded by Sagdeev [8]; he demonstrated the possibility of the existence of such waves and stressed the important role of dispersion effects, which lead to the formation of shock waves having an oscillatory structure. A detailed study of steady and nonsteady compression waves in a two-fluid plasma with allowance for dispersion effects is given in [9], but energy dissipation and thermal conduction are absent there and it is assumed that $\gamma = 2$. The nature of the singular points of the equations describing the structure of a shock wave propagating across a magnetic field is studied in [10] and the critical parameters at which a transition occurs from continuous solutions with respect to all the functions to discontinuous solutions with respect to the gasdynamic functions are established from this analysis. In [11] the different structures of shock waves transverse to the magnetic field are analyzed with allowance for the finite conductivity, dispersion, and electron thermal conductivity, numerical solutions are obtained for the nonlinear equations of the structure in the region of parameters when the flow is continuous, and the critical parameters at which the solution changes from an oscillatory to a monotonic or discontinuous solution are found. A rather detailed study of the isomagnetic compression shock is made in [12] using the numerical solution of the nonsteady problem of the propagation of shock waves in a plasma across a magnetic field (the program for the numerical solution was developed and tested on a large number of such nonsteady problems by Berezin; see [13], for example). Within the framework of this program an attempt was made in [14] to determine the critical parameters of shock waves propagating at an arbitrary angle to the undisturbed magnetic field in the absence of thermal conduction. However, because of the complex nature of the dependence of the width of the shock wave on the Mach number, which will be described in detail below, the critical Mach numbers obtained in [14] must be considered as very approximate. A table of values of the critical Mach numbers for a number of angles is given in [15] without an indication of the method of solution.

1. Initial Equations

The system of equations describing the one-dimensional motion of a two-fluid quasineutral plasma at an arbitrary angle to an undisturbed magnetic field with allowance for dispersion, finite conductivity, and electron thermal conductivity, has the following form in dimensionless variables:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0; \tag{1.1}$$

$$\frac{\partial}{\partial t}(nu) + \frac{\partial}{\partial x}\left(nu^2 + \frac{H^2 + B^2 + p}{2(1+\beta)}\right) = 0;$$

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$$\frac{\partial}{\partial t}(nv)+\frac{\partial}{\partial x}\left(nuv-\frac{H\sin\theta}{1+\beta}\right)=0;\quad \frac{\partial}{\partial t}(nw)+\frac{\partial}{\partial x}\left(nuw-\frac{B\sin\theta}{1+\beta}\right)=0;$$

$$\frac{\partial}{\partial t}\left[H-\frac{\beta}{1+\beta}\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial H}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[uH-v\sin\theta-\frac{\varkappa}{n}\frac{\partial H}{\partial x}-\frac{1-\beta}{1+\beta}\frac{\sin\theta}{n}\frac{\partial B}{\partial x}-\frac{\beta u}{1+\beta}\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial H}{\partial x}\right)\right]=0;$$

$$\frac{\partial}{\partial t}\left[B-\frac{\beta}{1+\beta}\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial B}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[uB-w\sin\theta-\frac{\varkappa}{n}\frac{\partial B}{\partial x}+\frac{1-\beta}{1+\beta}\frac{\sin\theta}{n}\frac{\partial H}{\partial x}-\frac{\beta u}{1+\beta}\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial B}{\partial x}\right)\right]=0;$$

$$\frac{\partial}{\partial t}\left\{\frac{p}{2(\gamma-1)}+\frac{1}{2}(H^2+B^2)+\frac{1+\beta}{2}n(u^2+v^2+w^2)+\right.$$
$$\left.+\frac{\beta}{2(1+\beta)n}\left[\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial B}{\partial x}\right)^2\right]\right\}+\frac{\partial}{\partial x}\left\{u\left[\frac{\gamma p}{2(\gamma-1)}+H^2+B^2+\right.\right.$$
$$\left.+\frac{1+\beta}{2}n(u^2+v^2+w^2)+\frac{\beta}{2(1+\beta)n}\left(\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial B}{\partial x}\right)^2\right)\right]-$$
$$-(vH+wB)\sin\theta-\frac{\varkappa}{n}\left(H\frac{\partial H}{\partial x}+B\frac{\partial B}{\partial x}\right)-\frac{1-\beta}{1+\beta}\frac{\sin\theta}{n}\left(H\frac{\partial B}{\partial x}-B\frac{\partial H}{\partial x}\right)-$$
$$\left.-\frac{\beta}{1+\beta}\left[H\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\left(\frac{1}{n}\frac{\partial H}{\partial x}\right)+B\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\left(\frac{1}{n}\frac{\partial B}{\partial x}\right)\right]-2\chi\frac{\partial T_e}{\partial x}\right\}=0,$$

$p = nT_e$, $\vec{u} = (m_i\vec{u}_i + m_e\vec{u}_e)/(m_i + m_e) = \{u, v, w\}$, $\vec{H} = \{\sin\theta, H, B\}$. The direction of wave propagation coincides with the x axis and the undisturbed magnetic field lies in the plane x, z. We take the quantities

$$n_0,\ \delta=\frac{c\sqrt{m_i}}{\sqrt{4\pi n_0}\,e},\ V_A=\frac{H_0}{\sqrt{4\pi n_0 m_i}},\ H_0,\ \frac{H_0^2}{8\pi}.$$

as the scales of density, length, velocity, magnetic field, and pressure. In addition, we introduce the following notation: $\varkappa = m_e c\nu/eH_0$ is the dimensionless effective collision frequency; $\beta = m_e/m_i$; θ is the angle between the plane of the wave front and the propagation direction; $\chi = [p/(H^2 + B^2 + \sin^2\theta)]\cdot[((\sin^2\theta\cdot\chi_{\parallel})/\varkappa)+(((H^2+B^2)\varkappa\chi_{\perp})/(H^2+B^2+\sin^2\theta))]H^2 + B^2 + \sin^2\theta)$ is the dimensionless coefficient of thermal conductivity (the coefficients along and across the magnetic field are designated by the subscripts $\parallel$ and $\perp$). The conductivity of the plasma in dimensionless variable was introduced through the equation $\sigma = ne^2/m_e\nu$.

2. Equations of Shock-Wave Structure

Let us change to a coordinate system moving with the constant velocity M of the shock wave in the direction of negative values of the coordinate x. Then from (1.1) we obtain the following system of steady-state equations and algebraic relations (conservation laws):

$$\begin{gathered} nu=\mathrm{M},\ 2\mathrm{M}u+p+H^2+B^2=C_1,\\ \mathrm{M}v=H\sin\theta,\ \mathrm{M}w=(B-\cos\theta)\sin\theta,\\ \beta\mathrm{M}\frac{d}{dx}\left(\frac{1}{n}\frac{dH}{dx}\right)=\mathrm{M}H-\varkappa\frac{dH}{dx}-\left(\frac{dB}{dx}+nv\right)\sin\theta,\\ \beta\mathrm{M}\frac{d}{dx}\left(\frac{1}{n}\frac{dB}{dx}\right)=\mathrm{M}B-\left(nw-\frac{dH}{dx}\right)\sin\theta-\varkappa\frac{dB}{dx}-\mathrm{M}n\cos\theta, \end{gathered} \tag{2.1}$$

$$\begin{gathered}\chi\frac{p-2\mathrm{M}u}{\mathrm{M}}\frac{du}{dx}=\frac{u}{\mathrm{M}}\left\{((2\chi-\varkappa)H+B\sin\theta)\frac{dH}{dx}+\right.\\ \left.+((2\chi-\varkappa)B-H\sin\theta)\frac{dB}{dx}\right\}+\frac{1}{2}\mathrm{M}(u^2+v^2+w^2)+u\left(\frac{\gamma p}{2(\gamma-1)}+\right.\\ \left.+H^2+B^2\right)-(vH+wB)\sin\theta-\beta u\left[H\frac{d}{dx}\left(\frac{1}{n}\frac{dH}{dx}\right)+B\frac{d}{dx}\left(\frac{1}{n}\frac{dB}{dx}\right)\right]-C_2,\end{gathered} \tag{2.2}$$

where

$$C_1 = 2\mathrm{M}^2 + p_0 + \cos^2\theta;\ C_2 = 0.5\mathrm{M}(\gamma p_0/(\gamma-1) + \mathrm{M}^2 + 2\cos^2\theta).$$

Equations (2.2) for the magnetic field components H and B and the longitudinal macroscopic velocity of the plasma together with the conservation laws (2.1) describe the structure of a standing shock wave with allowance for the inertia of the electrons (terms proportional to β), the finite conductivity, the anisotropy of the plasma, and electron thermal conductivity.

3. Steady States

We introduce the subscripts 0 and 1 to mark the functions corresponding to the steady states of the plasma in front of the wave (undisturbed state) and behind the wave (disturbed state), and from the system (2.1), (2.2) we obtain the following relation between these functions:

$$n_1u_1 = \mathrm{M},\ p_1 + 2\mathrm{M}u_1 + B_1^2 = p_0 + 2\mathrm{M}^2 + \cos^2\theta,$$
$$\mathrm{M}w_1 = (B_1 - \cos\theta)\sin\theta,\ n_1(\mathrm{M}\cos\theta + w_1\sin\theta) = \mathrm{M}B_1,$$
$$\mathrm{M}\left(\frac{\gamma p_1}{2(\gamma-1)n_1} + \frac{B_1^2}{n_1} + \frac{u_1^2+w_1^2}{2}\right) - w_1B_1\sin\theta = \mathrm{M}\left(\frac{\gamma p_0}{2(\gamma-1)} + \cos^2\theta + \frac{1}{2}\mathrm{M}^2\right).$$

It is known that the functions behind the shock wave are single-valued if the density n_1 behind the wave is taken as the independent variable. Therefore, expressing all the functions behind the wave through n_1 and the Mach number M, we obtain the following expression:

$$B_1 = (\mathrm{M}^2 - \sin^2\theta)\cos\theta \Big/ \left(\frac{\mathrm{M}^2}{n_1} - \sin^2\theta\right),\ u_1 = Mn_1^{-1},$$
$$p_1 = p_0 + 2\mathrm{M}^2(n_1 - 1)\left[\frac{1}{n_1} - \frac{(n_1+1)\mathrm{M}^2 - 2n_1\sin^2\theta}{2(\mathrm{M}^2 - n_1\sin^2\theta)^2}\cos^2\theta\right].$$

The Mach number of the shock wave is connected with the plasma density n_1 behind the wave by the equation

$$(\mathrm{M}^2 - n_1\sin^2\theta)^2\left(\mathrm{M}^2 - \frac{\gamma p_0 n_1}{\gamma+1-(\gamma-1)n_1}\right) - \left[\frac{(2-\gamma)n_1+\gamma}{\gamma+1-(\gamma-1)n_1}\mathrm{M}^2 - n_1\sin^2\theta\right]n_1\mathrm{M}^2\cos^2\theta = 0. \qquad (3.1)$$

If the new functions

$$y = \mathrm{M}^2/\sin^2\theta,\ z = y/n_1$$

are introduced, then from (3.1) we obtain

$$y = \frac{[(\gamma+1)(z-1)^2\sin^2\theta - (\gamma z - \gamma - 1)\cos^2\theta]\,z - \gamma(z-1)^2 p_0}{(\gamma-1)(z-1)^2\sin^2\theta + [(2-\gamma)z + \gamma - 1]\cos^2\theta}. \qquad (3.2)$$

By assigning different values of z one can find the values of y and then all the unknown functions

$$\mathrm{M}^2 = y\sin^2\theta,\ n_1 = y/z,$$
$$B_1 = [(y-1)/(z-1)]\cos\theta,$$
$$p_1 = p_0 + 2(y-z)\{1 - [(y+z-2)/2(z-1)^2]\operatorname{ctg}^2\theta\}\sin^2\theta.$$

For the case of $\gamma = 5/3$ we have from (3.2)

$$y = \frac{[8(z-1)^2\sin^2\theta - (5z-8)\cos^2\theta]\,z - 5(z-1)^2 p_0}{2(z-1)^2\sin^2\theta + (z+2)\cos^2\theta}.$$

Everywhere below we will consider states of the plasma in front of the wave with $p_0 \ll 1$, and therefore the subject will only be fast shock waves with $\mathrm{M} > 1$.

4. Study of Singular Points

The steady states 0 and 1 considered above correspond to singular points of the system (2.2), and the nature of the solution of the shock-wave problem depends on the type of these singular points. Taking $u = u_{0,1} + u'$, $B = B_{0,1} + B'$, and $H = H'$, where $u_0 = \mathrm{M}$, $B_0 = \cos\theta$, $u' \ll u_{0,1}$, and $B' \ll B_{0,1}$, we perform a linearization of the system (2.2) near the singular points 0 and 1. The allowance for dispersion does not affect the values of the critical parameters and therefore we will take $\beta = 0$; thus, we obtain from (2.2) the following linear system of equations (the primes are omitted):

$$\begin{gathered} c\,dH/dx = \varkappa a_{1,0}H - a_{1,0}\sin\theta\cdot B - b_{1,0}\sin\theta\cdot u, \\ c\,dB/dx = a_{1,0}\sin\theta\cdot H + \varkappa a_{1,0}B + \varkappa b_{1,0}u, \\ f_{1,0}\,du/dx = g_{1,0}H + r_{1,0}B + q_{1,0}u, \end{gathered} \qquad (4.1)$$

where

$$\begin{gathered} a_{1,0} = \mathrm{M}(1 - n_{1,0}\sin^2\theta/\mathrm{M}^2);\ c = \varkappa^2 + \sin^2\theta; \\ b_{1,0} = n_{1,0}^2\left(\cos\theta + \frac{B_{1,0} - \cos\theta}{\mathrm{M}^2}\sin^2\theta\right); \\ f_{1,0} = [(p_{1,0} - 2\mathrm{M}u_{1,0})/\mathrm{M}]\chi_{1,0};\ g_{1,0} = (2u_{1,0}B_{1,0}a_{1,0}\sin\theta)/\mathrm{M}c)\chi_{1,0}; \\ r_{1,0} = B_{1,0}\left[\frac{a_{1,0}u_{1,0}}{\mathrm{M}c}(2\chi_{1,0}\varkappa - \varkappa^2 - \sin^2\theta) - \frac{2-\gamma}{\gamma-1}u_{1,0} - \frac{\sin^2\theta}{\mathrm{M}}\right]; \\ q_{1,0} = u_{1,0}\left[\frac{B_{1,0}b_{1,0}}{\mathrm{M}c}(2\chi_{1,0}\varkappa - \varkappa^2 - \sin^2\theta) - \frac{\mathrm{M}}{\gamma-1}\right] + \frac{\gamma p_{1,0}}{2(\gamma-1)} + B_{1,0}^2; \\ \chi_{1,0} = \frac{p_{1,0}}{B_{1,0}^2 + \sin^2\theta}\left(\frac{\chi_\parallel}{\varkappa}\sin^2\theta + \frac{B_{1,0}^2\varkappa}{B_{1,0}^2 + \sin^2\theta}\chi_\perp\right). \end{gathered} \qquad (4.2)$$

As usual, we will seek a solution of (4.1) in the form of functions proportional to exp(kx), as a result of which we obtain the characteristic equation

$$c^2 f_{1,0} k^3 - c(2a_{1,0} f_{1,0} \varkappa + c q_{1,0}) k^2 + \left[\varkappa(\varkappa a_{1,0}^2 f_{1,0} + 2a_{1,0} c q_{1,0} - - b_{1,0} c r_{1,0}) + (b_{1,0} c g_{1,0} + a_{1,0}^2 f_{1,0} \sin\theta)\sin\theta\right] k - a_{1,0} c (a_{1,0} q_{1,0} - b_{1,0} r_{1,0}) = 0. \quad (4.3)$$

Let us consider the case of $\gamma = 5/3$ and assume that the thermal conduction can be neglected ($\chi = 0$). Then Eq. (4.3) becomes a quadratic equation:

$$c q_{1,0} k^2 - \varkappa(2a_{1,0} q_{1,0} - b_{1,0} r_{1,0}) k + a_{1,0}(a_{1,0} q_{1,0} - b_{1,0} r_{1,0}) = 0 \quad (4.4)$$

with the coefficients

$$r_{1,0} = -B_{1,0}[(3/2 - n_{1,0}\sin^2\theta/\mathrm{M}^2) u_{1,0} + \sin^2\theta/\mathrm{M}],$$

$$q_{1,0} = \frac{5}{4} p_{1,0} + B_{1,0}^2 - \mathrm{M} u_{1,0}\left(\frac{3}{2} + \frac{B_{1,0} b_{1,0}}{\mathrm{M}^2}\right),$$

where $a_{1,0}$, $b_{1,0}$, and c are the same as in (4.2).

In the undisturbed state

$$a_0 = \mathrm{M}(1 - \sin^2\theta/\mathrm{M}^2) > 0, \quad b_0 = \cos\theta > 0,$$

$$r_0 = -(3/2)\mathrm{M}\cos\theta < 0, \quad q_0 = -(3/2)M^2 < 0.$$

Therefore, the Routh sequence (see [16], for example)

$$\{c q_0, - \varkappa(2a_0 q_0 - b_0 r_0),\ a_0(a_0 q_0 - b_0 r_0)\}$$

for Eq. (4.4) at the undisturbed point 0 has the signs $\{-, +, -\}$ for any parameters. According to the Routh criterion, these two sign changes correspond to two roots having a positive real part, and the integral curve of the system (2.2) can always go to the point 0 (remember that the undisturbed state of the plasma is taken at $x < 0$).

In the undisturbed state

$$a_1 = \mathrm{M}(1 - n_1\sin^2\theta/\mathrm{M}^2) > 0,$$

$$b_1 = n_1^2\{\cos\theta + [(B_1 - \cos\theta)/\mathrm{M}^2]\sin^2\theta\} > 0,$$

$$r_1 = -(3/2) B_1 u_1 < 0,$$

$$q_1 = (5/4) p_1 + B_1^2 - \mathrm{M} u_1 (3/2 + B_1 b_1/\mathrm{M}^2)$$

and the number of sign changes in the Routh sequence depends on the sign of the quantity q_1. If $q_1 > 0$, then we have $\{+, -, +\}$, which corresponds to two roots having a positive real part at the singular point 1, and the integral curve cannot emerge from the point 1 (remember that the disturbed state of the plasma is taken at $x > 0$). If $q_1 < 0$, then we have $\{-, -, +\}$, which corresponds to the presence of one negative root, and the integral curve can emerge from the point 1. Thus, the transition from continuous solutions to discontinuous solutions is accomplished with parameters such that $q_1 = 0$. These critical parameters are presented in Table 1.

When thermal conduction is taken into account it is necessary to study the number of sign changes in the Routh sequence

$$\{A,\ B,\ C - AD/B,\ D\}$$

for Eq. (4.3), where A, B, C, and D are the coefficients of Eq. (4.3) in the order of decrease in the power of k. At the singular point 0 we have $\{-, +, -, +\}$, these three sign changes correspond to three roots having a positive real part, and the integral curve can always go to the point 0. At the singular point 1, as the analysis shows, the Routh sequence has the signs $\{-, -, +, -\}$ for Mach numbers less than some value $M_*(\theta)$ and $\{+, -, +, -\}$ for $M > M_*(\theta)$, i.e., in the first case the integral curve can emerge from the point 1, since there is one negative root of the characteristic equation (4.3), while in the second case it cannot, since Eq. (4.3) does not have a single root with a negative real part. This transition can occur when the coefficient f_1 is reduced to zero. The critical parameters obtained in this way in the presence of thermal conduction are presented in Table 2. The allowance for thermal conduction leads to an increase in the critical Mach number for all angles θ, with this difference decreasing with an increase in the angle θ. It is interesting to note that, as in the case of shock waves propagating across a magnetic field and analyzed in [10-12], for oblique waves with the critical parameters we have $p_1 = 2Mu_1$ or (in dimensional variables) $u_1 = (T_1/m_i)^{1/2}$, i.e., the longitudinal velocity of the plasma behind the wave (in the coordinate system connected with the wave) equals the ionic sound velocity behind the wave.

TABLE 1

$\theta°$	0	10	20	30	40	50	60	70	80	90
M	2,76	2,741	2,677	2,575	2,439	2,277	2,097	1,907	1,716	1,530
n_1	2,66	2,661	2,654	2,641	2,623	2,596	2,558	2,506	2,435	2,342
B_1	2,66	2,638	2,564	2,445	2,286	2,096	1,882	1,657	1,432	1,217
p_1	3,42	3,388	3,241	3,011	2,722	2,397	2,060	1,740	1,451	1,20
T_1	1,28	1,273	1,221	1,140	1,038	0,923	0,806	0,696	0,596	0,512

TABLE 2

$\theta°$	0	10	20	30	40	50	60	70	80	90
M	3,464	3,431	3,334	3.177	2,970	2,724	2,454	2,175	1,904	1,652
n_1	3,00	2,999	2,994	2,986	2,974	2,954	2,926	2,884	2,821	2,732
B_1	3,00	2,968	2,875	2,723	2,523	2,282	2,015	1,737	1,463	1,209
p_1	8,00	7,852	7,423	6,759	5,933	5,024	4,116	3,282	2,570	2,00
T_1	2,667	2,618	2,479	2,263	1,995	1,700	1,407	1,138	0,911	0,732

5. Engaged Shock Waves

Shock waves which propagate along the undisturbed magnetic field ($\theta = 90°$), which are called engaged shock waves, will be considered separately. In this case from the system (3.1) we have

$$\begin{gathered} n_1 = M^2,\ u_1 = M^{-1}, \\ p_1 = p_0 + 2(M^2 - 1) - B_1^2, \\ [\gamma p_0/(\gamma - 1) + M^2]M^2 = \gamma p_1/(\gamma - 1) + 1 + B_1^2. \end{gathered} \tag{5.1}$$

Assuming as before that $p_0 \ll 1$ and $\gamma = 5/3$, we obtain a simple relation between the magnetic field behind the wave and the Mach number (or the density):

$$B_1^2 = (2/3)(5M^2 - M^4 - 4) = (2/3)(5n_1 - n_1^2 - 4). \tag{5.2}$$

In the absence of thermal conduction, as for the oblique waves, the transition from continuous to discontinuous solutions occurs at critical parameters which are determined from the equation

$$q_1 = (5/4)p_1 - (3/2) = 0,$$

from which, using (5.1) and (5.2), we obtain the equation for the critical Mach number M_*:

$$5M_*^4 - 10M_*^2 - 4 = 0.$$

Thus, for engaged shock waves with $\chi = 0$ the critical parameters are

$$M_* = (1 + 3/\sqrt{5})^{1/2} \approx 1.53,\quad n_{1*} = 1 + 3/\sqrt{5} \approx 2.34,\quad B_{1*} = [1.2(\sqrt{5} - 1)]^{1/2} \approx 1.22,\quad p_{1*} = 1.2.$$

If thermal conduction is taken into account ($\chi \neq 0$), then the number of sign changes in the Routh sequence depends on the sign of the coefficient $f_1 = \chi_1(p_1 - 2Mu_1)/M$ in the characteristic equation. The critical Mach number is determined from the equation $M_*^4 - 2M_*^2 - 2 = 0$, and the critical parameters for engaged shock waves with allowance for thermal conduction are

$$M_* = (1 + \sqrt{3})^{1/2} \approx 1.65,\quad n_{1*} = 1 + \sqrt{3} \approx 2.73,\quad B_{1*} = [2(\sqrt{3} - 1)]^{1/2} \approx 1.21,\quad p_{1*} = 2.$$

6. Numerical Solution of the Structure Equations

The main purpose of the calculations is to obtain the structure of the shock waves with parameters close to the critical parameters. As the critical parameters are approached the profiles of the magnetic field and the temperature have a rather large width, which is determined by the values of $\varkappa$ and χ, while the width of the profiles of density and longitudinal velocity is sharply decreased. This circumstance leads to

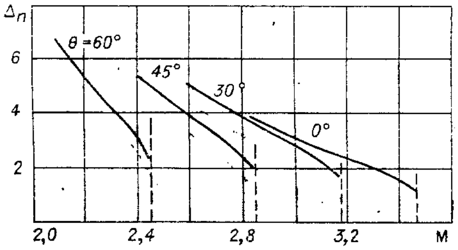

Fig. 1

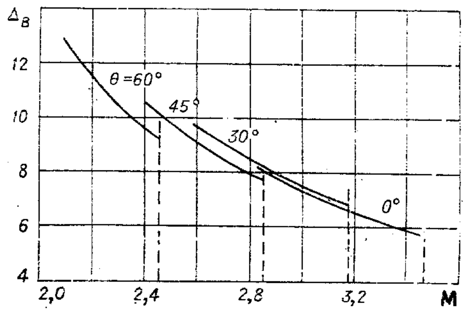

Fig. 2

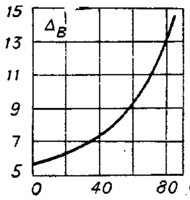

Fig. 3

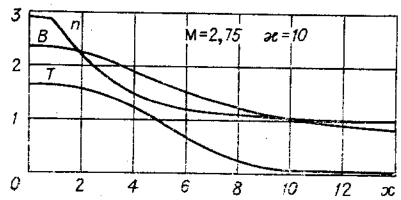

Fig. 4

certain difficulties. Therefore, a method is proposed for the numerical solution of the structure equations which can be used in the entire region of variation of the parameters up to the critical values.

Taking $\beta = 0$ in Eqs. (2.2), we obtain the following system of equations, which is then solved numerically:

$$dH/dx = E(H, B, u), \; dB/dx = F(H, B, u), \; du/dx = G(H, B, u), \tag{6.1}$$

where

$$E = (1/c)[\varkappa(\mathrm{M}H - nv\sin\theta) - (\mathrm{M}B - nw\sin\theta - \mathrm{M}n\cos\theta)\sin\theta];$$
$$F = (1/c)[(\mathrm{M}H - nv\sin\theta)\sin\theta + \varkappa(\mathrm{M}B - nw\sin\theta - \mathrm{M}n\cos\theta)];$$
$$G = [\mathrm{M}/\chi(p - 2\mathrm{M}u)]\{(uE/\mathrm{M})[(2\chi - \varkappa)H + B\sin\theta] + (uF/\mathrm{M})[(2\chi -$$
$$-\varkappa)B - H\sin\theta] + (1/2)\mathrm{M}(u^2 + v^2 + w^2) + u[(5/4)p + H^2 + B^2] - (vH + wB)\sin\theta - C_2\}.$$

The neglect of the dispersion terms, which are connected with the inertia of the electrons and are proportional to β, is possible in the case of $0 \le \theta \lesssim \sqrt{\beta}$ only with large enough values of $\varkappa$ (high dissipation), while in the case of $\theta > \sqrt{\beta}$ it is always possible.

In Eqs. (6.1) let us change from the spatial argument x to the arc length s of the function u(x) in accordance with the formula $ds = \sqrt{1 + (du/dx)^2}\,dx$. Then in place of the system (6.1) we obtain the following system:

$$dH/ds = E(1 + G^2)^{-1/2}, \; dB/ds = F(1 + G^2)^{-1/2}, \; du/ds = G(1 + G^2)^{-1/2},$$
$$dx/ds = (1 + G^2)^{-1/2}.$$

The change to the argument s allows us to use any of the methods of solving ordinary differential equations with a constant integration step.

As follows from the analysis of singular points (Sec. 4), the numerical solution of the structure equations must start from the vicinity of the singular point 1, taking as the initial conditions the solution of the linear system (4.1) corresponding to the single negative root of the characteristic equation (4.3) at the point 1.

The results of the solution of the system of equations (6.2) are presented in Figs. 1-5. The dependence of the width of the density profile $\Delta_n = (n_1 - 1)/|dn/dx|_{max}$ and of the magnetic field profile $\Delta_B = (B_1 - \cos\theta)/|dB/dx|_{max}$ on the Mach number for different angles θ is given in Figs. 1 and 2. The nature of the variation in the width of the density profile is qualitatively the same – its sharp decrease occurs only in the vicinity of the critical Mach number $M_*(\theta)$. The width Δ_B of the magnetic field profile approaches a certain limit Δ_B^*, which increases with an increase in $\varkappa$ and θ, as $M \to M_*(\theta)$. The dependence $\Delta_B^*(\theta)$ for a fixed value of $\varkappa$ is presented in Fig. 3. Profiles of the density, magnetic field, and temperature in a shock wave propagating at an angle $\theta = 45°$ for different states of the plasma behind the wave are presented in Figs. 4 and 5. It is seen from these graphs that the steepness of the density profile increases as $M \to M_*$.

As shown by an analysis of the structure equations with $M < M_*$, depending on the collision frequency $\varkappa$ the shock wave can have a monotonic structure (with $\varkappa \ge \varkappa_*$) or an oscillatory structure (with $\varkappa < \varkappa_*$), where

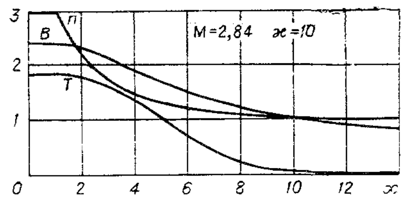

Fig. 5

$$\varkappa_* = \left[\frac{4(M^2-1)(M^2-\sin^2\theta)\sin^2\theta}{(2M^2-1-\sin^2\theta)^2-4(M^2-1)(M^2-\sin^2\theta)}\right]^{1/2}.$$

The nature of the singular point 0 changes when the discriminant of Eq. (4.4) changes sign; the formula for $\varkappa_*$ is obtained from this condition. Hence, it follows that the collision frequency $\varkappa_*$ increases with an increase in the Mach number M and in the angle θ.

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